BIGRADED BETTI NUMBERS OF SOME SIMPLE POLYTOPES

IVAN LIMONCHENKO

Abstract. The bigraded Betti numbers $\beta^{-i,2j}(P)$ of a simple polytope P are the dimensions of the bigraded components of the Tor groups of the face ring $\mathbf{k}[P]$. The numbers $\beta^{-i,2j}(P)$ reflect the combinatorial structure of P as well as the topology of the corresponding moment-angle manifold $\mathcal{Z}_P$, and therefore they find numerous applications in combinatorial commutative algebra and toric topology. Here we calculate some bigraded Betti numbers of the type $\beta^{-i,2(i+1)}$ for associahedra, and relate the calculation of the bigraded Betti numbers for truncation polytopes to the topology of their moment-angle manifolds. These two series of simple polytopes provide conjectural extrema for the values of $\beta^{-i,2j}(P)$ among all simple polytopes P with the fixed dimension and number of facets.

1. Introduction

We consider *simple convex n-dimensional polytopes* P in the Euclidean space $\mathbb{R}^n$ with scalar product $\langle\ ,\ \rangle$. Such a polytope P can be defined as an intersection of m halfspaces:

$$P=\big\{\boldsymbol{x}\in\mathbb{R}^n\colon \langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i\geqslant 0\quad\text{for } i=1,\dots,m\big\}, \tag{1.1}$$

where $\boldsymbol{a}_i\in\mathbb{R}^n$, $b_i\in\mathbb{R}$. We assume that the hyperplanes defined by the equations $\langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i=0$ are in general position, that is, at most n of them meet at a single point. We also assume that there are no redundant inequalities in (1.1), that is, no inequality can be removed from (1.1) without changing P. Then P has exactly m *facets* given by

$$F_i=\big\{\boldsymbol{x}\in P\colon \langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i=0\big\},\quad\text{for } i=1,\dots,m.$$

Let A_P be the $m\times n$ matrix of row vectors $\boldsymbol{a}_i$, and let $\boldsymbol{b}_P$ be the column vector of scalars $b_i\in\mathbb{R}$. Then we can write (1.1) as

$$P=\big\{\boldsymbol{x}\in\mathbb{R}^n\colon A_P\boldsymbol{x}+\boldsymbol{b}_P\geqslant\mathbf{0}\big\},$$

and consider the affine map

$$i_P\colon\mathbb{R}^n\to\mathbb{R}^m,\quad i_P(\boldsymbol{x})=A_P\boldsymbol{x}+\boldsymbol{b}_P.$$

It embeds P into

$$\mathbb{R}^m_{\geqslant}=\{\boldsymbol{y}\in\mathbb{R}^m\colon y_i\geqslant 0\quad\text{for } i=1,\dots,m\}.$$

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Following [3, Constr. 7.8], we define the space $\mathcal{Z}_P$ from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\
\Big\downarrow & & \Big\downarrow{\scriptstyle\mu} \\
P & \xrightarrow{i_P} & \mathbb{R}^m_\geqslant
\end{array} \tag{1.2}
$$

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{\boldsymbol{z} \in \mathbb{C}^m \colon |z_i| = 1 \quad \text{for } i = 1, \ldots, m\}$$

on $\mathbb{C}^m$. Therefore, $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient P, and i_Z is a $\mathbb{T}^m$-equivariant embedding.

By [3, Lemma 7.2], $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$, called the *moment-angle manifold* corresponding to P.

Denote by K_P the boundary ∂P^* of the dual simplicial polytope. It can be viewed as a simplicial complex on the set $[m] = \{1, \ldots, m\}$, whose simplices are subsets $\{i_1, \ldots, i_k\}$ such that $F_{i_1} \cap \ldots \cap F_{i_k} \neq \varnothing$ in P.

Let $\mathbf{k}$ be a field, let $\mathbf{k}[v_1, \ldots, v_m]$ be the graded polynomial algebra on m variables, $\deg(v_i) = 2$, and let $\Lambda[u_1, \ldots, u_m]$ be the exterior algebra, $\deg(u_i) = 1$. The *face ring* (also known as the *Stanley–Reisner ring*) of a simplicial complex K on $[m]$ is the quotient ring

$$\mathbf{k}[K] = \mathbf{k}[v_1, \ldots, v_m]/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_k}$ for which $\{i_1, \ldots, i_k\}$ is not a simplex in K. We refer to $\mathcal{I}_K$ as the *Stanley–Reisner ideal* of K.

Note that $\mathbf{k}[K]$ is a module over $\mathbf{k}[v_1, \ldots, v_m]$ via the quotient projection. The dimensions of the bigraded components of the Tor-groups,

$$\beta^{-i,2j}(K) := \dim_{\mathbf{k}} \operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[K], \mathbf{k}\bigr), \quad 0 \leqslant i, j \leqslant m,$$

are known as the *bigraded Betti numbers* of $\mathbf{k}[K]$, see [8] and [3, §3.3]. They are important invariants of the combinatorial structure of K. We denote

$$\beta^{-i,2j}(P) := \beta^{-i,2j}(K_P).$$

The Tor-groups and the bigraded Betti numbers acquire a topological interpretation by means of the following result on the cohomology of $\mathcal{Z}_P$:

Theorem 1.1 ([3, Theorem 8.6] or [6, Theorem 4.7])**.** *The cohomology algebra of the moment-angle manifold $\mathcal{Z}_P$ is given by the isomorphisms*

$$\begin{aligned} H^*(\mathcal{Z}_P; \mathbf{k}) &\cong \operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K_P], \mathbf{k}) \\ &\cong H\bigl[\Lambda[u_1, \ldots, u_m] \otimes \mathbf{k}[K_P], d\bigr], \end{aligned}$$

where the latter algebra is the cohomology of the differential bigraded algebra whose bigrading and differential are defined by

$$\operatorname{bideg} u_i = (-1, 2), \ \operatorname{bideg} v_i = (0, 2); \quad du_i = v_i, \ dv_i = 0.$$

Therefore, cohomology of $\mathcal{Z}_P$ acquires a bigrading and the topological Betti numbers $b^q(\mathcal{Z}_P) = \dim_k H^q(\mathcal{Z}_P; \mathbf{k})$ satisfy

$$b^q(\mathcal{Z}_P) = \sum_{-i+2j=q} \beta^{-i,2j}(P). \tag{1.3}$$

Poincaré duality in cohomology of $\mathcal{Z}_P$ respects the bigrading:

Theorem 1.2 ([3, Theorem 8.18])**.** *The following formula holds:*

$$\beta^{-i,2j}(P) = \beta^{-(m-n)+i,2(m-j)}(P).$$

From now on we shall drop the coefficient field $\mathbf{k}$ from the notation of (co)homology groups. Given a subset $I \subset [m]$, we denote by K_I the corresponding *full subcomplex* of K (the restriction of K to I). The following classical result can be also obtained as a corollary of Theorem 1.1:

Theorem 1.3 (Hochster, see [3, Cor. 8.8])**.** *Let $K = K_P$. We have:*

$$\beta^{-i,2j}(P) = \sum_{J \subset [m], |J|=j} \dim \widetilde{H}^{j-i-1}(K_J).$$

We also introduce the following subset in the boundary of P:

$$P_I = \bigcup_{i \in I} F_i \subset P. \tag{1.4}$$

Note that if $K = K_P$ then K_I is a deformation retract of P_I for any I. The following is a direct corollary of Theorem 1.3.

Corollary 1.4. *We have*

$$\beta^{-i,2(i+1)}(P) = \sum_{I \subset [m], |I|=i+1} \big(cc(P_I) - 1\big),$$

where $cc(P_I)$ is the number of connected components of the space P_I.

The structure of this paper is as follows. Calculations for Stasheff polytopes (also known as associahedra) are given in Section 2. In Section 3 we calculate the bigraded Betti numbers of truncation polytopes (iterated vertex cuts of simplices) completely. These calculations were first made in [10] using a similar but slightly different method; an alternative combinatorial argument was given in [4]. We also compare the calculations of the Betti numbers with the known description of the diffeomorphism type of $\mathcal{Z}_P$ for truncation polytopes [1].

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2. Stasheff polytopes

Stasheff polytopes, also known as *associahedra*, were introduced as combinatorial objects in the work of Stasheff on higher associativity [9]. Explicit convex realizations of Stasheff polytopes were found later by Milnor and others, see [2] for details.

We denote the n-dimensional Stasheff polytope by As^n. The i-dimensional faces of As^n ($0 \leqslant i \leqslant n-1$) bijectively correspond to the sets of $n-i$ pairwise

nonintersecting diagonals in an $(n+3)$-gon G_{n+3}. (We assume that diagonals having a common vertex are nonintersecting.) A face H belongs to a face H' if and only if the set of diagonals corresponding to H contains the set of diagonals corresponding to H'.

In particular, vertices of As^n correspond to complete triangulations of G_{n+3} by its diagonals, and facets of As^n correspond to diagonals of G_{n+3}. We therefore identify the set of diagonals in G_{n+3} with the set of facets $\{F_1, \ldots, F_m\}$ of As^n, and identify both sets with $[m]$ when it is convenient. Note that $m = \frac{n(n+3)}{2}$.

We shall need a convex realization of As^n from [2, Lecture II, Th. 5.1]:

Theorem 2.1. *As^n can be identified with the intersection of the parallelepiped*

$$\bigl\{\boldsymbol{y} \in \mathbb{R}^n \colon 0 \leqslant y_j \leqslant j(n+1-j) \quad \textit{for } \; 1 \leqslant j \leqslant n\bigr\}$$

with the halfspaces

$$\bigl\{\boldsymbol{y} \in \mathbb{R}^n \colon y_j - y_k + (j-k)k \geqslant 0\bigr\}$$

for $1 \leqslant k < j \leqslant n$.

Proposition 2.2. *We have:*

$$b^3(\mathcal{Z}_{As^n}) = \beta^{-1,4}(As^n) = \binom{n+3}{4}.$$

Proof. The number $\beta^{-1,4}(P)$ is equal to the number of monomials $v_i v_j$ in the Stanley–Reisner ideal of P [3, §3.3], or to the number of pairs of disjoint facets of P. In the case $P = As^n$ the latter number is equal to the number of pairs of intersecting diagonals in the $(n+3)$-gon G_{n+3}, see [2, Lecture II, Cor 6.2]. It remains to note that, for any 4-element subset of vertices of G_{n+3} there is a unique pair of intersecting diagonals whose endpoints are these 4 vertices. $\square$

Remark. The above calculation can be also made using the general formula $\beta^{-1,4}(P) = \binom{f_0}{2} - f_1$, see [3, Lemma 8.13], where f_i is the number of $(n-i-1)$-faces of P. The numbers f_i for As^n are well-known, see [2, Lecture II].

In what follows, we assume that there are no multiple intersection points of the diagonals of G_{n+3}, which can be achieved by a small perturbation of the vertices. We choose a cyclic order of vertices of G_{n+3}, so that 2 consequent vertices are joined by an edge. We refer to the diagonals of G_{n+3} joining the ith and the $(i+2)$th vertices (modulo $n+3$), for $i = 1, \ldots, n+3$ as *short*; other diagonals are *long*.

We refer to intersection points of diagonals inside G_{n+3} as *distinguished points*. A diagonal segment joining two distinguished points is called a *distinguished segment*. Finally, a *distinguished triangle* is a triangle whose vertices are distinguished points and whose edges are distinguished segments.

Theorem 2.3. *We have:*

$$b^4(\mathcal{Z}_{As^n}) = \beta^{-2,6}(As^n) = 5\binom{n+4}{6}$$

Proof. We need to calculate the number of generators in the 4th cohomology group of $H[\Lambda[u_1, \ldots, u_m] \times k[As^n], d]$, see Theorem 1.1 (note that here $m = \frac{(n+3)n}{2}$ is the number of diagonals in G_{n+3}). This group is generated by the cohomology classes of cocycles of the type $u_i u_j v_k$, where $i \neq j$ and $u_i v_k$, $u_j v_k$ are 3-cocycles. These 3-cocycles correspond to the pairs $\{i, k\}$ and $\{j, k\}$ of intersecting diagonals in G_{n+3}, or to a pair of distinguished points on the kth diagonal. It follows that every cocycle $u_i u_j v_k$ is represented by a distinguished segment. The identity

$$d(u_i u_j u_k) = u_i u_j v_k - u_i v_j u_k + v_i u_j u_k$$

implies that the cohomology classes represented by the cocycles in the right hand side are linearly dependent. Every such identity corresponds to a distinguished triangle.

We therefore obtain that $\beta^{-2,6}(As^n) = S_{n+3} - T_{n+3}$ where S_{n+3} is the number of distinguished segments and T_{n+3} is the number of distinguished triangles inside G_{n+3}. These numbers are calculated in the next three lemmas.

Lemma 2.4. *The number of distinguished triangles in G_{n+3} is given by*

$$T_{n+3} = \binom{n+3}{6}$$

Proof. We note that there is only one distinguished triangle in a hexagon (see Fig. 1); and therefore every 6 vertices of G_{n+3} contribute one distinguished triangle. ☐

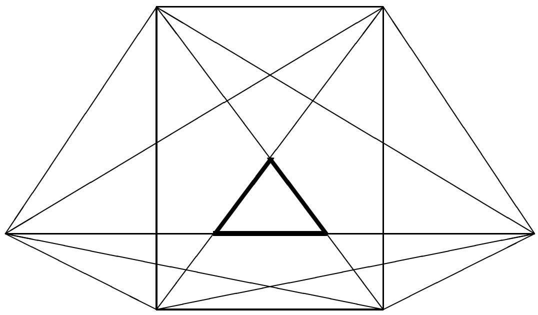

FIGURE 1.

Given a diagonal d of G_{n+3}, denote by $p(d)$ the number of distinguished points on d. We define the *length* of d as the smallest of the numbers of vertices of G_{n+3} in the open halfplanes defined by d. Therefore, short diagonals have length 1 and all diagonals have length $\leqslant \frac{n+1}{2}$. We refer to diagonals of maximal length simply as *maximal*. Obviously $p(d)$ depends only on the length of d, and we denote by $p(j)$ the number of distinguished points on a diagonal of length j.

Lemma 2.5. *If $n = 2k - 1$ is odd, then*

$$S_{n+3} = \frac{n+3}{2} \sum_{l=1}^{k-1} \Big(4l^2k^2 - 2k(2l^3 + l) \Big) + \frac{n+3}{4} k^2(k^2 - 1).$$

If $n = 2k-2$ is even, then

$$S_{n+3} = \frac{n+3}{2}\sum_{l=1}^{k-1}\Big(4l^2k^2 - 2k(2l^3+2l^2+l) + (l^4+2l^3+2l^2+l)\Big).$$

Proof. First assume that $n = 2k-1$. Then

$$S_{n+3} = \sum_d \frac{p(d)(p(d)-1)}{2} =$$

$$= (n+3)\left(\sum_{j=1}^{\frac{n+1}{2}} \frac{p(j)(p(j)-1)}{2}\right) - \left(\frac{n+3}{2}\right)\frac{p(\frac{n+1}{2})(p(\frac{n+1}{2})-1)}{2},$$

since the number of distinguished segments on the maximal diagonals is counted in the sum twice.

We denote by v the $(n+3)$th vertex of G_{n+3} and numerate the diagonals coming from v by their lengthes. We denote by $c(i,j)$ the number of intersection points of the jth diagonal coming from v with the diagonals from the ith vertex, for $1 \leqslant i \leqslant j \leqslant \frac{n+1}{2}$, and set $c(i,j) = 0$ for $i > j$. Then we have

$$p(j) = \sum_{i=1}^{\frac{n+1}{2}} c(i,j), \tag{2.1}$$

To compute $c(i,j)$ we note that

$$c(1,1) = n;$$
$$c(i,j-1) = c(i,j)+1 \quad \text{for } 1 \leqslant i < j \leqslant \frac{n+1}{2};$$
$$c(i+1,j+1) = c(i,j)-1 \quad \text{for } 1 \leqslant i \leqslant j \leqslant \frac{n-1}{2}.$$

It follows that

$$c(i,j) = c(1,j-i+1)-(i-1) = c(1,1)-(j-i)-(i-1) = n-j+1, \tag{2.2}$$

for $i \leqslant j$. Note that $c(i,j)$ does not depend on i. Substituting this in (2.1) and then substituting the resulting expression for $p(j)$ in the sum for S_{n+3} above we obtain the required formula.

The case $n = 2k-2$ is similar. The only difference is that there are two maximal diagonals coming from every vertex of G_{n+3}, so that no subtraction is needed in the sum for S_{n+3}. □

Lemma 2.6. *The number of distinguished segments is given by*

$$S_{n+3} = (n+3)\binom{n+3}{5}.$$

Proof. This follows from Lemma 2.5 by summation using the following formulae for the sums Σ_n of the nth powers of the first $(k-1)$ natural numbers:

$$\Sigma_1 = \frac{k(k-1)}{2}, \qquad \Sigma_2 = \frac{k(k-1)(2k-1)}{6},$$
$$\Sigma_3 = \frac{k^2(k-1)^2}{4}, \qquad \Sigma_4 = \frac{k(k-1)(2k-1)(3k^2-3k-1)}{30}.$$

□

Now Theorem 2.3 follows from Lemma 2.5 and Lemma 2.6. □

The following fact follows from the description of the combinatorial structure of As^n (see also [2, Lecture II, Cor. 6.2]):

Proposition 2.7. *Two facets F_1 and F_2 of the polytope As^n do not intersect if and only if the corresponding diagonals d_1 and d_2 of the polygon G_{n+3} intersect (in a distinguished point).*

Lemma 2.8. *The number of distinguished points on a maximal diagonal of G_{n+3} is given by*

$$q = q(n) = \begin{cases} \frac{n(n+2)}{4}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The case $n = 2$ is obvious. If n is odd, then setting $j = \frac{n+1}{2}$ in (2.1) and using (2.2) we calculate

$$p(j) = \sum_{i=1}^{\frac{n+1}{2}} c\Big(i, \frac{n+1}{2}\Big) = \frac{(n+1)^2}{4}.$$

If n is even, then the maximal diagonal has length $j = \frac{n}{2}$. It is easy to see that we have $p(j) = \sum_{i=1}^{n/2} c(i,j)$ instead of (2.1), and (2.2) still holds. Therefore,

$$p(j) = \sum_{i=1}^{\frac{n}{2}} c\Big(i, \frac{n}{2}\Big) = \frac{n(n+2)}{4}.$$

□

Theorem 2.9. *Let $P = As^n$ be an n-dimensional associahedron, $n \geqslant 3$. The bigraded Betti numbers of P satisfy*

$$\beta^{-q,2(q+1)}(P) = \begin{cases} n+3, & \text{if } n \text{ is even;} \\ \frac{n+3}{2}, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta^{-i,2(i+1)}(P) = 0 \quad \text{for } i \geqslant q+1;$$

where $q = q(n)$ is given in Lemma 2.8.

Proof. We prove the theorem by induction on n. The base case $n = 3$ can be seen from the tables of bigraded Betti numbers below. By Corollary 1.4, in order to calculate $\beta^{-i,2(i+1)}(P)$, we need to find all $I \subset [m]$, $|I| = i+1$, whose corresponding P_I has more than one connected component. In the case $i = q$ we shall prove that $cc(P_I) \leqslant 2$ for $|I| = q+1$, and describe explicitly those I for which $cc(P_I) = 2$. In the case $i > q$ we shall prove that $cc(P_I) = 1$ for $|I| = i+1$. These statements will be proven as separate lemmas; the step of induction will follow at the end.

We numerate the vertices of G_{n+3} by the integers from 1 to $n+3$. Then every diagonal d corresponds to an ordered pair (i,j) of integers such that $i < j-1$. It is convenient to view the diagonal corresponding to (i,j) as the segment $[i,j]$ inside the segment $[1, n+3]$ on the real line. Then Proposition 2.7 may be reformulated as follows:

Proposition 2.10. *The facets F_1 and F_2 of $P = As^n$ do not intersect if and only if the corresponding segments $[i_1, j_1]$ and $[i_2, j_2]$ overlap, that is,*

$$F_1 \cap F_2 = \varnothing \quad \Longleftrightarrow \quad i_1 < i_2 < j_1 < j_2 \quad \textit{or} \quad i_2 < i_1 < j_2 < j_1.$$

Let I be a set of diagonals of G_{n+3} (or integer segments in $[1, n+3]$), and P_I the corresponding set (1.4). We write $I = I_1 \sqcup I_2$ whenever P_I has exactly two connected components corresponding to I_1 and I_2. We also denote by $e(I)$ the set of endpoints of segments from I; its a subset of integers between 1 and $n+3$.

Proposition 2.11. *If $I = I_1 \sqcup I_2$ then the subsets $e(I_1)$ and $e(I_2)$ are disjoint.*

Proof. Follows directly from Proposition 2.10. □

Given an integer $m \in [1, n+3]$ and a set of segments I, we denote by $c_I(m)$ the number of segments in I that have m as one of their endpoints (equivalently, the number of diagonals in I with endpoint m). Then $0 \leqslant c_I(m) \leqslant n$.

Proposition 2.12. *If $I = I_1 \sqcup I_2$ then there exists m such that $c_I(m) \leqslant \frac{n+1}{2}$.*

Proof. Assume the opposite is true. Choose integers $m_1 \in e(I_1)$ and $m_2 \in e(I_2)$. Since $c_I(m_1) > \frac{n+1}{2}$, $c_I(m_2) > \frac{n+1}{2}$ and $e(I_1)$, $e(I_2)$ are disjoint by the previous proposition, we obtain that the total number of elements in $e(I)$ is more than $2 + \frac{n+1}{2} + \frac{n+1}{2} = n+3$. A contradiction. □

Lemma 2.13. *We have that $cc(P_I) \leqslant 2$ for $|I| > l(n) = \frac{n(n+2)}{4}$.*

Proof. We prove this lemma by induction on n.

First let $n = 3$, and assume that the statement of the lemma fails, i.e. there is a set $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup \ldots$ of diagonals of G_6, $|I| \geqslant 4$, such that $cc(P_I) \geqslant 3$. As there are only 3 long diagonals in G_6, there exists a short diagonal $d \in I$; assume $d \in I_1$. Since $cc(P_I) \geqslant 3$, every $e \in I_2$ and $f \in I_3$ intersect d. Hence, e and f meet at a vertex A of G_6. This contradicts the fact that $e(I_2)$ and $e(I_3)$ are disjoint (see Proposition 2.11).

Now let $n > 3$ and assume that there is a set $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup \ldots$ of diagonals of G_{n+3}, $|I| > \frac{n(n+2)}{4}$, with $cc(P_I) \geqslant 3$.

If there exists $m \in [1, n+3]$ with $c_I(m) = 0$, then we may assume that m is the first vertex, and view I as a set of diagonals of G_{n+2} (the segment $[2, n+3]$ cannot belong to I, since otherwise $cc(P_I) = 1$). As $l(n) > l(n-1)$, the induction assumption finishes the proof of the lemma.

Now $c_I(m) \geqslant 1$ for every $m \in [1, n+3]$. Then by the argument similar to that of Proposition 2.12, there exists m with $c_I(m) \leqslant \frac{n}{3}$. Consider 2 cases:

1. There exists $m_0 \in e(I_k)$ for some $1 \leqslant k \leqslant cc(P_I)$ with the smallest value of $c_I(m) \leqslant \frac{n}{3}$, such that $|I_k| > c_I(m_0)$.

We may assume that one of these m_0 is the first vertex. Removing from I all segments with endpoint 1, we obtain a new set $\tilde{I}$ of segments inside $[2, n+3]$ (the segment $[2, n+3]$ cannot belong to I, as otherwise $cc(P_I) \leqslant 2$). We have:

$$|\tilde{I}| = |I| - c_I(1) > \frac{n(n+2)}{4} - \frac{n}{3} > \frac{(n-1)(n+1)}{4} = l(n-1).$$

By the induction assumption, $2 \geqslant cc(P_{\tilde{I}}) \geqslant cc(P_I) \geqslant 3$. A contradiction.

2. For every vertex m_0 with the smallest value of $c_I(m) \geqslant 1$ we have $|I_k| = c_I(m_0)$, where $m_0 \in e(I_k)$.

Again, we may assume that one of these m_0 is the first vertex $1 \in I_k$. We have $c_I(1) = 1$, as otherwise there are $\geqslant 2$ integer points m inside $[2, n+3]$ which belong to $e(I_k)$ and have $c_I(m) = 1$ (remember that $|I_k| = c_I(m_0)$).

Without loss of generality we may assume that $k = 1$. Then

$$|I| = 1 + |I_2| + |I_3| + \ldots \leqslant 1 + (1 + q(n-1)) \leqslant 2 + \frac{n^2}{4} \leqslant \frac{n(n+2)}{4}.$$

The first inequality above holds since $\tilde{I} = I_2 \sqcup I_3 \sqcup \ldots$ is a set of diagonals of G_{n+2} (the segment $[2, n+3]$ cannot belong to I, because $cc(P_I) \geqslant 3$), and we can apply to $\tilde{I}$ the induction assumption in the proof of the main Theorem 2.9, which gives us $|\tilde{I}| \leqslant 1 + q(n-1)$. We get a contradiction with the assumption $|I| > \frac{n(n+2)}{4}$. □

Lemma 2.14. *Assume that $I = I_1 \sqcup I_2,\ \ |I| \geqslant q+1,\ \ |I_1| \geqslant 2$ and $|I_2| \geqslant 2$. Then there exists another I' such that $I' = I'_1 \sqcup I'_2,\ \ |I'_1| = 1$ and $|I'| > |I|$.*

Proof. The proof is by induction on n. The cases $n = 3, 4, 5$ are checked by a direct computation (see also the tables at the end of this section).

Changing the numeration of vertices of G_{n+3} if necessary, we may assume that the first vertex has the smallest value of $c_I(m)$. Then $c_I(1) \leqslant \frac{n+1}{2}$ by Proposition 2.12. Without loss of generality we may assume that $1 \notin e(I_1)$.

We claim that the segment $[2, n+3]$ does not belong to I. Indeed, in the opposite case $c_I(1) > 0$ (otherwise $cc(P_I) = 1$), $1 \in e(I_2)$, $[2, n+3] \in I_1$. If $c_I(1) \geqslant 2$, then there is an integer point $m \in e(I_2)$ inside $[2, n+3]$ with $c_I(m) = 1 < c_I(1)$, which contradicts the choice of the first vertex. Then $c_I(1) = 1$ and $[2, n+3] \in I_1$ imply that $|I_2| = c_I(1) = 1$ which contradicts the assumption $|I_2| \geqslant 2$ in the lemma.

Removing from I all segments with endpoint 1, we obtain a new set $\tilde{I}$ of integer segments inside $[2, n+3]$. Note that

$$|\tilde{I}| = |I| - c_I(1) \geqslant |I| - \left[\tfrac{n+1}{2}\right]. \tag{2.3}$$

We want to apply the induction assumption to the set $\tilde{I}$ of integer segments inside $[2, n+3]$, viewed as diagonals in an $(n+2)$-gon G_{n+2}. To do this, we need to check the assumptions of the lemma for $\tilde{I}$.

First, we claim that $\tilde{I} = \tilde{I}_1 \sqcup \tilde{I}_2$, i.e. $P_{\tilde{I}}$ has exactly two connected components. Indeed, it obviously has at least two components, and the number of components cannot be more than two by Lemma 2.13, since

$$|\tilde{I}| \geqslant |I| - \frac{n+1}{2} \geqslant q + 1 - \frac{n+1}{2} > \frac{(n+1)^2}{4} - \frac{n+1}{2} = l(n-1).$$

Second, $|\tilde{I}_1| = |I_1| \geqslant 2$ and $|I_2| \geqslant |\tilde{I}_2| \geqslant 1$. If $|\tilde{I}_2| = 1$ then we have either $c_I(1) = 1$ or $c_I(1) = 2$. (Indeed, if $c_I(1) = 0$ then $|I_2| = |\tilde{I}_2| = 1$, which contradicts the assumption, and $c_I(1)$ cannot be more than 2 as otherwise $c_I(1)$ is not the smallest one.) Therefore, $|I_2| \leqslant 3$. We also have $|I_1| = |\tilde{I}_1| \leqslant p(d)$,where $d \in \tilde{I}_2 = \{d\}$, because d intersects every diagonal from I_1. Due to Lemma 2.8, $p(d) \leqslant q(n-1) \leqslant \frac{n^2}{4}$. Hence,

$$|I| = |I_1| + |I_2| \leqslant p(d) + 3 \leqslant \frac{n^2}{4} + 3 \leqslant \frac{(n+1)^2}{4} < q(n) + 1 \leqslant |I|$$

for $n \geqslant 6$. A contradiction. Thus, $|\widetilde{I_2}| \geqslant 2$.

It remains to check that $|\tilde{I}| \geqslant q(n-1)+1$. If n is odd, then

$$|\tilde{I}| \geqslant |I| - \frac{n+1}{2} \geqslant \frac{(n+1)^2}{4} + 1 - \frac{n+1}{2} = \frac{(n-1)(n+1)}{4} + 1 = q(n-1)+1.$$

If n is even, then

$$|\tilde{I}| \geqslant |I| - \frac{n}{2} \geqslant \frac{n(n+2)}{4} + 1 - \frac{n}{2} = \frac{n^2}{4} + 1 = q(n-1)+1.$$

Now, applying the induction assumption to $\tilde{I}$, we find a new set of integer segments $\tilde{J}$ inside $[2, n+3]$ with $|\tilde{J}| > |\tilde{I}|$ and $|\tilde{J}_1| = 1$. Then $\tilde{J}_1 = \{d\}$, where d is a diagonal of G_{n+2}. Hence, $|\tilde{J}| = |\tilde{J}_1| + |\tilde{J}_2| \leqslant 1 + p(d)$. We have $p(d) \leqslant q(n-1)$, and the equality holds if and only if $d = d_{max}$ is a maximal diagonal in G_{n+2}. Therefore, we can replace $\tilde{J}$ by $J' = J_1' \sqcup J_2'$, where $J_1' = \{d_{max}\}$ and J_2' is the set of diagonals in G_{n+2} which intersect d_{max} at its distinguished points. Indeed, we have

$$|J'| = 1 + q(n-1) \geqslant 1 + p(d) \geqslant |\tilde{J}| > |\tilde{I}|. \tag{2.4}$$

Choosing d_{max} in G_{n+2} as the diagonal corresponding to the segment $[2, k]$ where $k = \left[\frac{n+7}{2}\right]$ we observe that it is also a maximal diagonal for G_{n+3}. Now take $I_1' = \{d_{max}\}$ and take I_2' to be the union of J_2' and all diagonals with endpoint 1 intersecting d_{max}. Since the number of distinguished points on d_{max} is $\left[\frac{n+1}{2}\right]$, we obtain from (2.4) and (2.3)

$$|I'| = 1 + |I_2'| = 1 + |J_2'| + \left[\tfrac{n+1}{2}\right] = |J'| + \left[\tfrac{n+1}{2}\right] > |\tilde{I}| + \left[\tfrac{n+1}{2}\right] \geqslant |I|,$$

which finishes the inductive argument. □

Lemma 2.15. *Suppose $cc(P_I) = 2$, $I = I_1 \sqcup I_2$ and $|I| \geqslant q+1$. Then either $|I_1| = 1$ or $|I_2| = 1$.*

Proof. Assume the opposite, i.e. $|I_1| \geqslant 2$ and $|I_2| \geqslant 2$. By Lemma 2.14, we may find another $I' = I_1' \sqcup I_2'$ such that $|I_1'| = 1$ and $|I'| > |I| \geqslant q+1$. On the other hand $|I_1'| = 1$ implies that $I_1' = \{d\}$ and $|I'| \leqslant 1 + p(d) \leqslant 1 + q$. A contradiction. □

Lemma 2.16. *Suppose $cc(P_I) = 2$, $I = I_1 \sqcup I_2$ and $|I| = q+1$. Then I_1 consists of a single maximal diagonal d_{max}, and I_2 consists of all diagonals of G_{n+3} which intersect d_{max}.*

Proof. By Lemma 2.15, we may assume that I_1 consists of a single diagonal d. Then

$$1 + q = |I| = |I_1| + |I_2| \leqslant 1 + p(d) \leqslant 1 + q,$$

which implies that $p(d) = q$ and $|I_2| = p(d)$. □

Lemma 2.17. *Suppose $|I| > q+1$. Then $cc(P_I) = 1$.*

Proof. We have $|I| > q+1 > l(n)$. Hence, $cc(P_I) \leqslant 2$ by Lemma 2.13. Assume $cc(P_I) = 2$ and $I = I_1 \sqcup I_2$. Then $|I_1| = 1$ by Lemma 2.15, i.e. $I_1 = \{d\}$ and $|I| \leqslant 1 + p(d) \leqslant 1 + q$. This contradicts the assumption $|I| > q+1$. □

Now we can finish the induction in the proof of Theorem 2.9. From Corollary 1.4 and Lemma 2.16 we obtain that the number $\beta^{-q,2(q+1)}(P)$ is equal to the number of maximal diagonals in G_{n+3}. The latter equals $n+3$ when n is even, and $\frac{n+3}{2}$ when n is odd. The fact that $\beta^{-i,2(i+1)}(P)$ vanishes for $i \geqslant q+1$ follows from Corollary 1.4 and Lemma 2.17. $\square$

We also calculate the bigraded Betti numbers of As^n for $n \leqslant 5$ using software package *Macaulay 2*, see [5].

The tables below have $n-1$ rows and $m-n-1$ columns. The number in the intersection of the kth row and the lth column is $\beta^{-l,2(l+k)}(As^n)$, where $1 \leqslant l \leqslant m-n-1$ and $2 \leqslant l+k \leqslant m-2$. The other bigraded Betti numbers are zero except for $\beta^{0,0}(As^n) = \beta^{-(m-n),2m}(As^n) = 1$, see [3, Ch.8]. The bigraded Betti numbers given by Theorem 2.9 are printed in bold.

1. $n=2$, $m=5$.

5	5

2. $n=3$, $m=9$.

15	35	24	**3**	**0**
0	**3**	24	35	15

3. $n=4$, $m=14$.

35	140	217	154	49	**7**	**0**	**0**	**0**
0	28	266	784	1094	784	266	28	0
0	**0**	**0**	**7**	49	154	217	140	35

4. $n=5$, $m=20$.

70	420	1089	1544	1300	680	226	44	**4**	**0**	**0**
0	144	1796	8332	20924	32309	32184	20798	8480	2053	264
0	0	12	264	2053	8480	20798	32184	32309	20924	8332
0	**0**	**0**	**0**	**0**	**4**	44	226	680	1300	1544

The topology of moment-angle manifolds $\mathcal{Z}_P$ corresponding to associahedra is far from being well understood even in the case when P is 3-dimensional. In this case the cohomology ring $H^*(\mathcal{Z}_P)$ has nontrivial triple *Massey products* by a result of Baskakov (see [3, §8.4] or [6, §5.3]), which implies that $\mathcal{Z}_P$ is not *formal* in the sense of rational homotopy theory.

3. Truncation polytopes

Let P be a simple n-polytope and $v \in P$ its vertex. Choose a hyperplane H such that H separates v from the other vertices and v belongs to the positive halfspace $H_{\geqslant}$ determined by H. Then $P \cap H_{\geqslant}$ is an n-simplex, and $P \cap H_{\leqslant}$ is a simple polytope, which we refer to as a *vertex cut* of P. When the choice of the cut vertex is clear or irrelevant we use the notation $vc(P)$. We also use the notation $vc^k(P)$ for a polytope obtained from P by iterating the vertex cut operation k times.

As an example of this procedure, we consider the polytope $vc^k(\Delta^n)$, where Δ^n is an n-simplex, $n \geqslant 2$. We refer to $vc^k(\Delta^n)$ as a *truncation polytope*; it has $m = n+k+1$ facets. Note that the combinatorial type of $vc^k(\Delta^n)$ depends on the choice of the cut vertices if $k \geqslant 3$, however we shall not reflect this in the notation.

Simplicial polytopes dual to $vc^k(\Delta^n)$ are known as *stacked polytopes*. They can be obtained from Δ^n by iteratively adding pyramids over facets.

The Betti numbers for stacked polytopes were calculated in [10], but the grading used there was different. We include this result below, with a proof that uses a slightly different argument and our 'topological' grading and notation:

Theorem 3.1. *Let $P = vc^k(\Delta^n)$ be a truncation polytope. Then for $n \geqslant 3$ the bigraded Betti numbers are given by the following formulae:*

$$\begin{aligned}
&\beta^{-i,2(i+1)}(P) = i\binom{k+1}{i+1},\\
&\beta^{-i,2(i+n-1)}(P) = (k+1-i)\binom{k+1}{k+2-i},\\
&\beta^{-i,2j}(P) = 0, \quad \textit{for } i+1 < j < i+n-1.
\end{aligned}$$

The other bigraded Betti numbers are also zero, except for

$$\beta^{0,0}(P) = \beta^{-(m-n),2m}(P) = 1.$$

Remark. The first of the above formulae was proved in [4] combinatorially.

Proof. We start by analysing the behavior of bigraded Betti numbers under a single vertex cut. Let P be an arbitrary simple polytope and $P' = vc(P)$. We denote by Q and Q' the dual simplicial polytopes respectively, and denote by K and K' their boundary simplicial complexes. Then Q' is obtained by adding a pyramid with vertex v over a facet F of Q. We also denote by V, V' and $V(F)$ the vertex sets of Q, Q' and F respectively, so that $V' = V \cup v$.

The proof of the first formula is based on the following lemma:

Lemma 3.2. *Let P be a simple n-polytope with m facets and $P' = vc(P)$. Then*

$$\beta^{-i,2(i+1)}(P') = \binom{m-n}{i} + \beta^{-(i-1),2i}(P) + \beta^{-i,2(i+1)}(P).$$

Proof. Applying Theorem 1.3 for $j = i+1$, we obtain:

$$\begin{aligned}
\beta^{-i,2(i+1)}(P') &= \sum_{W\subset V',\, |W|=i+1} \dim \widetilde{H}_0(K'_W)\\
&= \sum_{W\subset V',\, v\in W,\, |W|=i+1} \dim \widetilde{H}_0(K'_W) && (3.1)\\
&+ \sum_{W\subset V',\, v\notin W,\, |W|=i+1} \dim \widetilde{H}_0(K'_W). && (3.2)
\end{aligned}$$

Sum (3.2) above is $\beta^{-i,2(i+1)}(P)$ by Theorem 1.3.

For sum (3.1) we have: in W there are i 'old' vertices and one new vertex v. Therefore, the number of connected components of K'_W (which is by 1 greater than the dimension of $\widetilde{H}_0(K'_W)$) either remains the same (if $W \cap F \neq \varnothing$) or increases by 1 (if $W \cap F = \emptyset$, in which case the new component is the new vertex v). The number of subsets W of the latter type is equal to the

number of ways to choose i vertices from the $m-n$ 'old' vertices that do not lie in F. Sum (3.1) is therefore given by

$$\sum_{W\subset V,|W|=i} \dim \widetilde{H}_0(K_W) + \binom{m-n}{i} = \beta^{-(i-1),2i}(P) + \binom{m-n}{i},$$

where we used Theorem 1.3 again. □

Now the first formula of Theorem 3.1 follows by induction on the number of cut vertices, using the fact that $\beta^{-i,2(i+1)}(\Delta^n) = 0$ for all i and Lemma 3.2.

The second formula follows from the bigraded Poincare duality, see Theorem 1.2.

The proof of the third formula relies on the following lemma.

Lemma 3.3. *Let P be a truncation polytope, K the boundary complex of the dual simplicial polytope, V the vertex set of K, and W a nonempty proper subset of V. Then*

$$\widetilde{H}_i(K_W) = 0 \quad \text{for } i \neq 0, n-2.$$

Proof. The proof is by induction on the number $m = |V|$ of vertices of K. If $m = n+1$, then P is an n-simplex, and K_W is contractible for every proper subset $W \subset V$.

To make the induction step we consider $V' = V \cup v$ and $V(F)$ as in the beginning of the proof of Theorem 3.1. Assume the statement is proved for V and let W be a proper subset of V'.

We consider the following 5 cases.

Case 1: $v \in W$, $W \cap V(F) \neq \varnothing$.

If $V(F) \subset W$, then K'_W is a subdivision of $K_{W-\{v\}}$. It follows that $\widetilde{H}_i(K'_W) \cong \widetilde{H}_i(K_{W-\{v\}})$.

If $W \cap V(F) \neq V(F)$, then we have

$$K'_W = K_{W-\{v\}} \cup K'_{W\cap V(F)\cup\{v\}}, \quad K_{W-\{v\}} \cap K'_{W\cap V(F)\cup\{v\}} = K_{W\cap V(F)},$$

and both $K_{W\cap V(F)}$ and $K'_{W\cap V(F)\cup\{v\}}$ are contractible. From the Mayer–Vietoris exact sequence we again obtain $\widetilde{H}_i(K'_W) \cong \widetilde{H}_i(K_{W-\{v\}})$.

Case 2: $v \in W$, $W \cap V(F) = \varnothing$.

In this case it is easy to see that $K'_W = K_{W-\{v\}} \sqcup \{v\}$. It follows that

$$\widetilde{H}_i(K'_W) \cong \begin{cases} \widetilde{H}_i(K_{W-\{v\}}) \oplus \mathbf{k}, & \text{for } i = 0; \\ \widetilde{H}_i(K_{W-\{v\}}), & \text{for } i > 0. \end{cases}$$

Case 3: $W = V' - \{v\} = V$.

Then K'_W is a triangulated $(n-1)$-disk and therefore contractible.

Case 4: $v \notin W$, $V(F) \subset W$, $W \neq V$.

We have

$$K_W = K'_W \cup F, \quad K'_W \cap F = \partial F,$$

where ∂F is the boundary of the facet F. Since ∂F is a triangulated $(n-2)$-sphere and F is a triangulated $(n-1)$-disk, the Mayer–Vietoris homology sequence implies that

$$\widetilde{H}_i(K'_W) \cong \begin{cases} \widetilde{H}_i(K_W), & \text{for } i < n-2; \\ \widetilde{H}_i(K_W) \oplus \mathbf{k}, & \text{for } i = n-2. \end{cases}$$

Case 5: $v \notin W$, $V(F) \not\subset W$. In this case we have $K'_W \cong K_W$.

In all cases we obtain

$$\widetilde{H}_i(K'_W) \cong \widetilde{H}_i(K_{W-\{v\}}) = 0 \quad \text{for } 0 < i < n-2,$$

which finishes the proof by induction. □

Now the third formula of Theorem 3.1 follows from Theorem 1.3 and Lemma 3.3.

The last statement of Theorem 3.1 follows from [3, Cor. 8.19]. □

For the sake of completeness we include the calculation of the bigraded Betti numbers in the case $n = 2$, that is, when P is a polygon.

Proposition 3.4. *If $P = vc^k(\Delta^2)$ is an $(k+3)$-gon, then*

$$\begin{aligned} &\beta^{-i,2(i+1)}(P) = i\binom{k+1}{i+1} + (k+1-i)\binom{k+1}{k+2-i}, \\ &\beta^{0,0}(P) = \beta^{-(k+1),2(k+3)}(P) = 1, \\ &\beta^{-i,2j}(P) = 0, \quad \textit{otherwise}. \end{aligned}$$

Proof. This calculation was done in [3, Example 8.21]. It can be also obtained by a Mayer–Vietoris argument as in the proof of Theorem 3.1. □

Corollary 3.5. *The bigraded Betti numbers of truncation polytopes $P = vc^k(\Delta^n)$ depend only on the dimension and the number of facets of P and do not depend on its combinatorial type. Moreover the numbers $\beta^{-i,2(i+1)}$ do not depend on the dimension n.*

The topological type of the corresponding moment-angle manifold $\mathcal{Z}_P$ is described as follows:

Theorem 3.6 (see [1, Theorem 6.3])**.** *Let $P = vc^k(\Delta^n)$ be a truncation polytope. Then the corresponding moment-angle manifold $\mathcal{Z}_P$ is diffeomorphic to the connected sum of sphere products:*

$$\mathop{\#}_{j=1}^{k} \left(S^{j+2} \times S^{2n+k-j-1}\right)^{\# j\binom{k+1}{j+1}},$$

where $X^{\# k}$ denotes the connected sum of k copies of X.

It is easy to see that the Betti numbers of the connected sum above agree with the bigraded Betti numbers of P, see (1.3).

References

[1] Frédéric Bosio and Laurent Meersseman. *Real quadrics in* $\mathbb{C}^n$*, complex manifolds and convex polytopes.* Acta Math. **197** (2006), no. 1, 53–127.

[2] Victor M. Buchstaber. *Lectures on toric topology.* In *Proceedings of Toric Topology Workshop KAIST 2008.* Trends in Math. **10**, no. 1. Information Center for Mathematical Sciences, KAIST, 2008, pp. 1–64.

[3] Victor M. Buchstaber and Taras E. Panov. *Torus Actions in Topology and Combinatorics* (in Russian). MCCME, Moscow, 2004, 272 pages.

[4] Suyoung Choi and Jang Soo Kim. *A combinatorial proof of a formula for Betti numbers of a stacked polytope.* Electron. J. Combin. **17** (2010), no. 1, Research Paper 9, 8 pp.; arXiv:math.CO/0902.2444.

[5] *Macaulay 2.* A software system devoted to supporting research in algebraic geometry and commutative algebra. Available at `http://www.math.uiuc.edu/Macaulay2/`

[6] Taras Panov. *Cohomology of face rings, and torus actions*, in "Surveys in Contemporary Mathematics". London Math. Soc. Lecture Note Series, vol. **347**, Cambridge, U.K., 2008, pp. 165–201; arXiv:math.AT/0506526.

[7] Taras Panov. *Moment–angle manifolds and complexes.* In *Proceedings of Toric Topology Workshop KAIST 2010.* Trends in Math. **12**, no. 1. Information Center for Mathematical Sciences, KAIST, 2010, pp. 43–69.

[8] Richard P. Stanley. *Combinatorics and Commutative Algebra*, second edition. Progr. in Math. **41**. Birkhäuser, Boston, 1996.

[9] James D. Stasheff. *Homotopy associativity of H-spaces. I.* Transactions Amer. Math. Soc. **108** (1963), 275–292.

[10] Naoki Terai and Takayuki Hibi. *Computation of Betti numbers of monomial ideals associated with stacked polytopes.* Manuscripta Math., 92(4): 447–453, 1997.

Department of Geometry and Topology, Faculty of Mathematics and Mechanics, Moscow State University, Leninskiye Gory, Moscow 119992, Russia

E-mail address: `iylim@mail.ru`